Available online at www.sciencedirect.com
SCIENCE DIRECT®
LINEAR ALGEBRA AND ITS APPLICATIONS
ELSEVIER
Linear Algebra and its Applications 368 (2003) 269–278
www.elsevier.com/locate/laa

An improved upper bound for Laplacian graph eigenvalues

Kinkar ch. Das

Department of Mathematics, Indian Institute of Technology, Kharagpur 721302, West Bengal, India

Received 22 February 2001; accepted 8 November 2002

Submitted by R.A. Brualdi

Abstract

Let $G = (V, E)$ be a simple graph on vertex set $V = \{v_1, v_2, \ldots, v_n\}$. Further let d_i be the degree of v_i and N_i be the set of neighbors of v_i. It is shown that

$$\max\left\{d_i + d_j - |N_i \cap N_j| : 1 \leqslant i < j \leqslant n,\ v_i v_j \in E\right\}$$

is an upper bound for the largest eigenvalue of the Laplacian matrix of G, where $|N_i \cap N_j|$ denotes the number of common neighbors between v_i and v_j. For any G, this bound does not exceed the order of G.

Further using the concept of common neighbors another upper bound for the largest eigenvalue of the Laplacian matrix of a graph has been obtained as

$$\max\left\{\sqrt{2\left(d_i^2 + d_i m_i'\right)} : 1 \leqslant i \leqslant n\right\},$$

where

$$m_i' = \frac{\sum_j \left\{d_j - |N_i \cap N_j| : v_i v_j \in E\right\}}{d_i}.$$

AMS classification: 05c50

Keywords: Graph; Laplacian matrix

E-mail address: kinkar@maths.iitkgp.ernet.in (K.ch. Das).

0024-3795/03/$ - see front matter © 2003 Elsevier Science Inc. All rights reserved.
doi:10.1016/S0024-3795(02)00687-0

1. Introduction

Let $G = (V, E)$ be a simple graph on vertex set $V = \{v_1, v_2, v_3, \ldots, v_n\}$. For $v \in V$, the degree of v, the set of neighbors of v and the average of the degrees of the vertices adjacent to v are denoted by d_v, N_v and m_v respectively. Let $A(G)$ be the adjacency matrix of G and let $D(G)$ be the diagonal matrix of vertex degrees. The Laplacian matrix of G is $L(G) = D(G) - A(G)$. Clearly, $L(G)$ is a real symmetric matrix. From this fact and Gersgorin's theorem, it follows that its eigenvalues are nonnegative real numbers. Moreover since its rows sum to 0, 0 is the smallest eigenvalue of $L(G)$. In [1, Theorem 1, p. 143], it is proved that if λ is an eigenvalue of $L(G)$, then $\lambda \leqslant n$ and that the multiplicity of 0 equals the number of components of G.

Denote the eigenvalues of $L(G)$ by $\lambda_1 \geqslant \lambda_2 \geqslant \lambda_3 \geqslant \cdots \geqslant \lambda_n = 0$.

Among the known upper bounds for λ_1 are the following:

1. Anderson and Morley's bound [1]:
$$\lambda_1 \leqslant \max\{d_u + d_v : uv \in E\}. \tag{1}$$
2. Li and Zhang's bound [2]: If $d_1 \geqslant d_2 \geqslant d_3 \geqslant \cdots \geqslant d_n$ are the degrees of the vertices of G (here, we are not assuming that d_i is the degree of v_i), then
$$\lambda_1 \leqslant 2 + \sqrt{(d_1 + d_2 - 2)(d_1 + d_3 - 2)}. \tag{2}$$
3. Another Li and Zhang's bound [2]: If r is the right-hand side of (1), if $xy \in E$ is such that $d_x + d_y = r$ and if $s = \max\{d_u + d_v : uv \in E - \{xy\}\}$, then
$$\lambda_1 \leqslant 2 + \sqrt{(r-2)(s-2)}. \tag{3}$$
4. Merris's bound [3]:
$$\lambda_1 \leqslant \max\{d_u + m_u : u \in V\}. \tag{4}$$
5. A new Li and Zhang's bound [4]:
$$\lambda_1 \leqslant \max\left\{\frac{d_u(d_u + m_u) + d_v(d_v + m_v)}{d_u + d_v} : uv \in E\right\}. \tag{5}$$
6. Rojo et al. [5]:
$$\lambda_1 \leqslant \max\left\{d_i + d_j - |N_i \cap N_j| : 1 \leqslant i < j \leqslant n\right\}, \tag{6}$$
where d_i denotes the degree of v_i and $|N_i \cap N_j|$ is the number of common neighbors of v_i and v_j.

2. Upper bounds for the largest eigenvalue

Rojo [5] has pointed out that sometimes some of these bounds ((1)–(5)) can give trivial results, that is, results which are greater than n, but (6) gives always a nontrivial upper bound of λ_1. Rojo [5] has also mentioned that (1) is never better than (6), but we have seen that sometimes (6) is not better than (1).[1] The following theorem presents a bound which is always better than (1) and (6).

Theorem 2.1. *If G is a graph on vertex set $V = \{v_1, v_2, \ldots, v_n\}$, then*

$$\lambda_1 \leqslant \max\left\{d_i + d_j - |N_i \cap N_j| : 1 \leqslant i < j \leqslant n,\ v_i v_j \in E\right\}, \tag{7}$$

where d_i denotes the degree of v_i and $|N_i \cap N_j|$ is the number of common neighbors of v_i and v_j. This upper bound for λ_1 does not exceed n.

Proof. If G has no edges, both sides of Eq. (7) are zero. Otherwise, it suffices to prove the result for at least one edge in the graph G.

Let x_k, $k = 1, 2, 3, \ldots, n$ be the eigencomponents of the eigenvector $\mathbf{X}$ corresponding to the eigenvalue λ_1 of the Laplacian matrix L.

We can assume that one of the eigencomponents (say x_i) is equal to 1 and the other eigencomponents are less than or equal to 1 in magnitude i.e., $x_i = 1$ and $|x_k| \leqslant 1$ for all k. Also let $x_j = \min_k\{x_k : v_i v_k \in E\}$.

Let c_{ij} be the number of common neighbors of v_i and v_j. Therefore $c_{ij} = |N_i \cap N_j|$. Now, since $x_j \leqslant x_k$ for all k such that $v_i v_k \in E$,

$$\sum_k \left\{x_k : v_i v_k \in E \ \&\ v_j v_k \notin E\right\} \geqslant (d_i - c_{ij}) x_j$$

and since $x_k \leqslant 1$ for all k,

$$\sum_k \left\{x_k : v_j v_k \in E \ \&\ v_i v_k \notin E\right\} \leqslant (d_j - c_{ij}).$$

We have

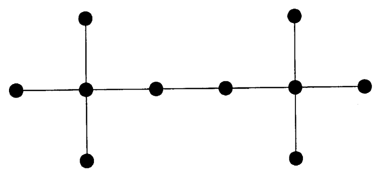
$$L\mathbf{X} = \lambda_1 \mathbf{X}. \tag{8}$$

From the ith equation of (8) we have

$$\lambda_1 x_i = d_i x_i - \sum_k \{x_k : v_i v_k \in E\}, \tag{9}$$

[1] For the following graph G the use of (1) and (6) gives $\lambda_1(G) \leqslant 6$ and $\lambda_1(G) \leqslant 8$ respectively.

G :



For this graph (1) is better than (6).

i.e.,

$$\lambda_1 = d_i - \sum_k \left\{x_k : v_i v_k \in E \ \& \ v_j v_k \in E\right\}$$
$$- \sum_k \left\{x_k : v_i v_k \in E \ \& \ v_j v_k \notin E\right\}. \tag{10}$$

From the jth equation of (8) we have

$$\lambda_1 x_j = d_j x_j - \sum_k \{x_k : v_j v_k \in E\},$$

i.e.,

$$\lambda_1 x_j = d_j x_j - \sum_k \left\{x_k : v_j v_k \in E \ \& \ v_i v_k \in E\right\}$$
$$- \sum_k \left\{x_k : v_j v_k \in E \ \& \ v_i v_k \notin E\right\}. \tag{11}$$

Subtracting (11) from (10), we get

$$\begin{aligned}\lambda_1(1 - x_j) = d_i - d_j x_j &- \sum_k \left\{x_k : v_i v_k \in E \ \& \ v_j v_k \notin E\right\} \\ &+ \sum_k \left\{x_k : v_j v_k \in E \ \& \ v_i v_k \notin E\right\} \\ &\leqslant d_i - d_j x_j - (d_i - c_{ij}) x_j + (d_j - c_{ij}) \\ &\leqslant (d_i + d_j - c_{ij})(1 - x_j). \end{aligned} \tag{12}$$

If $x_j = 1$, then $x_k = 1$ for all k such that $v_i v_k \in E$. Therefore from (9),

$$\lambda_1 = d_i - \sum_k \{x_k : v_i v_k \in E\} = d_i - d_i = 0.$$

But it is not possible for at least one edge in the graph. Therefore $x_j \neq 1$.

From (12) we get,

$$\lambda_1 \leqslant (d_i + d_j - c_{ij}),$$

where $v_i v_j \in E$. Hence

$$\lambda_1 \leqslant \max\left\{d_i + d_j - c_{ij} : 1 \leqslant i < j \leqslant n, \ v_i v_j \in E\right\},$$

i.e.,

$$\lambda_1 \leqslant \max\left\{d_i + d_j - |N_i \cap N_j| : 1 \leqslant i < j \leqslant n, \ v_i v_j \in E\right\}.$$

It is obvious that

$$\begin{aligned}&\max\left\{d_i + d_j - |N_i \cap N_j| : 1 \leqslant i < j \leqslant n, \ v_i v_j \in E\right\} \\ &\quad \leqslant \max\left\{d_i + d_j - |N_i \cap N_j| : 1 \leqslant i < j \leqslant n\right\}.\end{aligned}$$

In [5, Theorem 4, p. 157], it is proved that

$$\max\left\{d_i + d_j - |N_i \cap N_j| : 1 \leqslant i < j \leqslant n\right\} \leqslant n.$$

Therefore

$$\max\left\{d_i + d_j - |N_i \cap N_j| : 1 \leqslant i < j \leqslant n,\ v_i v_j \in E\right\} \leqslant n.$$

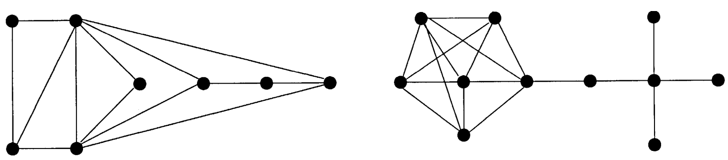
This proves that the bound given by (7) does not exceed n. □

Remark 2.2. It is easily seen that

$$\max\left\{d_i + d_j - |N_i \cap N_j| : 1 \leqslant i < j \leqslant n,\ v_i v_j \in E\right\}$$
$$\leqslant \max\{d_i + d_j : v_i v_j \in E\}.$$

Hence (7) is always better than (1).

Example 2.3.



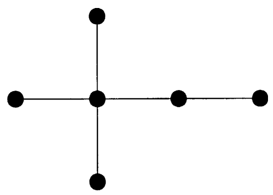
$G1$ $G2$

For these graphs $\lambda_1(G_1) = 7.1$ and $\lambda_1(G_2) = 6.57$, rounded to two decimal places and the mentioned bounds give the following results:

	(1)	(2)	(3)	(4)	(5)	(6)	(7)
G_1	11	9.94	9.94	9.00	8.73	8	8
G_2	10	10	10	9.40	9.40	9	7

Example 2.4. It has been already seen that (7) is always better than (1) and (6), but (7) is not always better than (2)–(5). For instance, for the graph G_3,

$G3$:



the use of (2)–(5) gives $\lambda_1(G_3) \leqslant 5.46$, $\lambda_1(G_3) \leqslant 5.46$, $\lambda_1(G_3) \leqslant 5.25$ and $\lambda_1(G_3) \leqslant 5.20$ respectively, while the use (7) gives $\lambda_1 \leqslant 6$.

Here is an another upper bound on eigenvalues of the Laplacian matrix of a graph. Sometimes this upper bound is better than (1)–(6).

Theorem 2.5. *If G is a graph on vertex set $V = \{v_1, v_2, \ldots, v_n\}$, then*

$$\lambda_1 \leqslant \max\left\{\sqrt{2\left(d_i^2 + d_i m_i'\right)} : 1 \leqslant i \leqslant n\right\}, \tag{13}$$

where

$$m_i' = \frac{\sum_j \left\{d_j - |N_i \cap N_j| : v_i v_j \in E\right\}}{d_i},$$

d_i denotes the degree of v_i and $|N_i \cap N_j|$ is the number of common neighbors of v_i and v_j.

Proof. Let x_k, $k = 1, 2, 3, \ldots, n$ be the eigencomponents of the eigenvector $\mathbf{X}$ corresponding to the eigenvalue λ_1 of the Laplacian matrix L.

We can assume that one of the eigencomponents (say x_i) is equal to 1 and the other eigencomponents are less than or equal to 1 in magnitude i.e., $x_i = 1$ and $|x_k| \leqslant 1$ for all k. Let c_{ij} be the number of common neighbors of v_i and v_j. Therefore $c_{ij} = |N_i \cap N_j|$.

Now, since $x_k \geqslant -1$ for all k,

$$\sum_k \left\{x_k : v_i v_k \in E \;\&\; v_j v_k \notin E\right\} \geqslant -(d_i - c_{ij})$$

and since $x_k \leqslant 1$ for all k,

$$\sum_k \left\{x_k : v_j v_k \in E \;\&\; v_i v_k \notin E\right\} \leqslant (d_j - c_{ij}).$$

We have

$$L\mathbf{X} = \lambda_1 \mathbf{X}. \tag{14}$$

From the ith equation of (14) we have

$$\lambda_1 x_i = d_i x_i - \sum_k \{x_k : v_i v_k \in E\},$$

i.e.,

$$\lambda_1 = d_i - \sum_k \left\{x_k : v_i v_k \in E \;\&\; v_j v_k \in E\right\} - \sum_k \left\{x_k : v_i v_k \in E \;\&\; v_j v_k \notin E\right\}. \tag{15}$$

From the jth equation of (14) we have

$$\begin{aligned}\lambda_1 x_j &= d_j x_j - \sum_k \left\{x_k : v_j v_k \in E\right\}\\ &= d_j x_j - \sum_k \left\{x_k : v_j v_k \in E \;\&\; v_i v_k \in E\right\}\\ &\quad - \sum_k \left\{x_k : v_j v_k \in E \;\&\; v_i v_k \notin E\right\}. \end{aligned} \tag{16}$$

Subtracting (16) from (15), we get

$$\begin{aligned}\lambda_1(1 - x_j) &= d_i - d_j x_j - \sum_k \left\{x_k : v_i v_k \in E \;\&\; v_j v_k \notin E\right\}\\ &\quad + \sum_k \left\{x_k : v_j v_k \in E \;\&\; v_i v_k \notin E\right\}\\ &\leqslant 2(d_i + d_j - c_{ij}). \end{aligned} \tag{17}$$

Taking summation over j on both sides of (17) such that $v_i v_j \in E$, we get

$$\lambda_1^2 \leqslant 2\left(d_i^2 + \sum_j \{d_j - c_{ij} : v_i v_j \in E\}\right) \leqslant 2\left(d_i^2 + d_i m_i'\right).$$

Therefore

$$\lambda_1 \leqslant \max\left\{\sqrt{2\left(d_i^2 + d_i m_i'\right)} : 1 \leqslant i \leqslant n\right\},$$

where

$$m_i' = \frac{\sum_j \left\{d_j - |N_i \cap N_j| : v_i v_j \in E\right\}}{d_i}. \qquad \square$$

Example 2.6. For graph G_2 in Example 2.3, upper bound given by (13) is 8.37 rounded to two decimal places. This bound is better than (1)–(6).

Theorem 2.7. *If G is a graph on vertex set $V = \{v_1, v_2, v_3, \dots, v_n\}$, then*

$$\lambda_1 \leqslant \max\left\{\sqrt{\begin{array}{l} d_i^2 + d_i + \sum_j \left\{d_i + d_j - |N_i \cap N_j| : v_i v_j \in E\right\}\\ \quad + \sum_j \left\{|N_i \cap N_j| : v_i v_j \notin E\right\}\end{array}} : 1 \leqslant i \leqslant n\right\}, \tag{18}$$

where d_i denotes the degree of v_i and $|N_i \cap N_j|$ is the number of common neighbors of v_i and v_j.

Proof. If G has no edges, both sides of Eq. (18) are zero. Otherwise, it suffices to prove the result for connected graphs. Let us consider the matrix L^2.

Now, (i, j)th element of L^2 is

$$\begin{cases} d_i^2 + d_i, & \text{if } i = j, \\ -d_i - d_j + |N_i \cap N_j|, & \text{if } v_i v_j \in E, \\ |N_i \cap N_j|, & \text{otherwise.} \end{cases}$$

Applying Gersgorin's theorem to the rows of L^2, we get

$$\lambda_1^2 \leqslant \max \Bigg\{ d_i^2 + d_i + \sum_j \big\{ d_i + d_j - |N_i \cap N_j| : v_i v_j \in E \big\} \\ + \sum_j \big\{ |N_i \cap N_j| : v_i v_j \notin E \big\} : 1 \leqslant i \leqslant n \Bigg\},$$

i.e.,

$$\lambda_1 \leqslant \sqrt{\begin{array}{l} \max \Big\{ d_i^2 + d_i + \sum_j \big\{ d_i + d_j - |N_i \cap N_j| : v_i v_j \in E \big\} \\ \qquad + \sum_j \big\{ |N_i \cap N_j| : v_i v_j \notin E \big\} : 1 \leqslant i \leqslant n \Big\}. \end{array}}$$

Therefore

$$\lambda_1 \leqslant \max \left\{ \sqrt{\begin{array}{l} d_i^2 + d_i + \sum_j \big\{ d_i + d_j - |N_i \cap N_j| : v_i v_j \in E \big\} \\ \quad + \sum_j \big\{ |N_i \cap N_j| : v_i v_j \notin E \big\} \end{array}} : 1 \leqslant i \leqslant n \right\}.$$

□

Theorem 2.8. *If G is a graph on vertex set $V = \{v_1, v_2, v_3, \ldots, v_n\}$, then*

$$\sum_j \big\{ |N_i \cap N_j| : j \neq i \big\} = \sum_j \big\{ (d_j - 1) : v_i v_j \in E \big\}, \quad v_i \in V, \tag{19}$$

where d_i be the degree of the vertex v_i and $|N_i \cap N_j|$ is the number of common neighbors of v_i and v_j.

Proof. Let us choose arbitrarily vertex v_i. For fixed v_i, let B_k be the set of edges e_{jk} such that $v_j v_k \in E$ and $v_i v_k \in E$, $j \neq i$, $v_k \in V$, i.e.,

$$B_k = \big\{ e_{jk} : v_j v_k \in E \ \& \ v_i v_k \in E, \ j \neq i \big\}, \quad v_k \in V.$$

Therefore $|B_k| = (d_k - 1)$. If $k \neq l$, then $B_k \cap B_l = \phi$. Let $B = \cup_k \{B_k : v_i v_k \in E\}$. Therefore

$$|B| = \sum_k \big\{ |B_k| : v_i v_k \in E \big\} = \sum_k \big\{ (d_k - 1) : v_i v_k \in E \big\}.$$

For fixed v_i, let D_k be the set of edges e_{kj} such that $v_jv_k \in E$ and $v_iv_j \in E$, $k \neq i$, $v_k \in V$, i.e.,

$$D_k = \left\{e_{kj} : v_jv_k \in E \;\&\; v_iv_j \in E\right\}, \quad k \neq i, \;\; v_k \in V.$$

Therefore $|D_k| = |N_i \cap N_k|$. If $k \neq l$, then $D_k \cap D_l = \phi$. Let $D = \bigcup_k \{D_k : k \neq i\}$. Therefore

$$|D| = \sum_k \left\{|D_k| : k \neq i\right\} = \sum_k \left\{|N_i \cap N_k| : k \neq i\right\}.$$

Next I shall prove that $B = D$. For this, let $e_{pq} \in B \leftrightarrow e_{pq} \in B_q \leftrightarrow v_pv_q \in E$ & $v_iv_q \in E$, $p \neq i \leftrightarrow e_{pq} \in D_p \leftrightarrow e_{pq} \in D$. Therefore $B = D$, i.e., $|B| = |D|$. Hence

$$\sum_j \left\{|N_i \cap N_j| : j \neq i\right\} = \sum_j \left\{(d_j - 1) : v_iv_j \in E\right\}.$$

Since v_i is arbitrary, the proof is finished. □

Remark 2.9. Basically the upper bounds given by (13) and (18) are identical. But method given by (13) is more efficient than (18). Now,

$$\begin{aligned}
& d_i^2 + d_i + \sum_j \left\{d_i + d_j - |N_i \cap N_j| : v_iv_j \in E\right\} \\
& \quad + \sum_j \left\{|N_i \cap N_j| : v_iv_j \notin E\right\} \\
& \quad = d_i^2 + d_i + \sum_j \left\{d_i + d_j - 2|N_i \cap N_j| : v_iv_j \in E\right\} \\
& \qquad + \sum_j \left\{|N_i \cap N_j| : j \neq i\right\} \\
& \quad = d_i^2 + d_i + \sum_j \left\{d_i + 2d_j - 1 - 2|N_i \cap N_j| : v_iv_j \in E\right\} \quad \text{(using (19))} \\
& \quad = 2d_i^2 + 2d_im_i' \quad \left(\text{since } m_i' = \frac{\sum_j \left\{d_j - |N_i \cap N_j| : v_iv_j \in E\right\}}{d_i}\right).
\end{aligned}$$

Hence, the upper bounds given by (13) and (18) are same.

Acknowledgements

The author would like to thank Dr. P. Kumar for his help and guidance. The author is grateful to the referee for his careful reading of the manuscript and correction of the grammatical mistakes.

References

[1] W.N. Anderson, T.D. Morley, Eigenvalues of the Laplacian of a graph, Linear and Multilinear Algebra 18 (1985) 141–145.
[2] J.-S. Li, D. Zhang, A new upper bound for eigenvalues of the Laplacian matrix of a graph, Linear Algebra Appl. 265 (1997) 93–100.
[3] R. Merris, A note on Laplacian graph eigenvalues, Linear Algebra Appl. 285 (1998) 33–35.
[4] J.-S. Li, D. Zhang, On Laplacian eigenvalues of a graph, Linear Algebra Appl. 285 (1998) 305–307.
[5] O. Rojo, R. Soto, H. Rojo, An always nontrivial upper bound for Laplacian graph eigenvalues, Linear Algebra Appl. 312 (2000) 155–159.